MATH 170A: Homework #3

Professor P.F. Rodriguez
Lecture 1
Assignment: 1, 2, 3 , 4; Supplemental: 15, 16, 18, 28, 30, 31
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Problem 1

You just moved into a new city in which 20% of cars are compact, 20% of cars are sport cars and 60% of cars are SUVs, and you want a buy a car to use it in the city. What car would you buy to maximize the chance of surviving in a collision (assume that collision of any pair of cars has the same probability).

Solution

Let

$$S_1 = \{\text{surviving in collision when driving a compact car}\}$$
$$S_2 = \{\text{surviving in collision when driving a sports car}\}$$
$$S_3 = \{\text{surviving in collision when driving an SUV}\}$$

Then, to answer the question, we calculate the probability of each of these and compare them to see which yields the greatest probability of surviving. Let $A_1 = \{\text{compact}\}, A_2 = \{\text{sports}\}, A_3 = \{\text{SUV}\}$. Then $\mathbf{P}(A_1) = 0.2, \mathbf{P}(A_2) = 0.2, \mathbf{P}(A_3) = 0.6$. Using the Total Probability law and calculating each of the probabilities, we get

$$\begin{aligned}
\mathbf{P}(S_1) &= \mathbf{P}(A_1)\mathbf{P}(S_1|A_1) + \mathbf{P}(A_2)\mathbf{P}(S_1|A_2) + \mathbf{P}(A_3)\mathbf{P}(S_1|A_3) \\
&= 0.2 \cdot 0.8 + 0.2 \cdot 0.9 + 0.6 \cdot 0.85 = 0.85 \\
\mathbf{P}(S_2) &= \mathbf{P}(A_1)\mathbf{P}(S_2|A_1) + \mathbf{P}(A_2)\mathbf{P}(S_2|A_2) + \mathbf{P}(A_3)\mathbf{P}(S_2|A_3) \\
&= 0.2 \cdot 0.8 + 0.2 \cdot 0.8 + 0.6 \cdot 0.7 = 0.74 \\
\mathbf{P}(S_3) &= \mathbf{P}(A_1)\mathbf{P}(S_3|A_1) + \mathbf{P}(A_2)\mathbf{P}(S_3|A_2) + \mathbf{P}(A_3)\mathbf{P}(S_3|A_3) \\
&= 0.2 \cdot 0.95 + 0.2 \cdot 0.9 + 0.6 \cdot 0.8 = 0.85
\end{aligned}$$

We see that the probability of surviving is highest for S_1 and S_3, both having 0.85 chance of surviving in a collision, so we can conclude that to maximize the probability, we should buy either a compact car or an SUV. ■

Problem 2

If a day is sunny the probability that the next day will be rainy is 1/2. If exactly k consecutive days have been rainy probability that the following day will be sunny is $\frac{1}{k+1}$. If today is sunny what is the probability that the next n days will all be rainy?

Solution

We're given that today is sunny, so we can denote today as day 0 so that indexing for the next n days is more convenient. Since today is sunny, the probability that the next day, day 1, will be rainy is $\frac{1}{2}$. Given that day 1 is rainy, we're also given that the probability that the following day is sunny is $\frac{1}{2}$. Equivalently, the probability that the following day is rainy is $1 - \frac{1}{2} = \frac{1}{2}$. Given day 1 and 2 are rainy, the probability that the following day is rainy is $1 - \frac{1}{1+2} = \frac{2}{3}$. We continue this for the n days and we see that given any day $k-1$ with all previous days raining, the probability that the kth day is rainy is $\frac{k-1}{k}$. Finally, to calculate that the probability that the next n days will all be rainy, we apply the multiplication rule and multiply the probabilities

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n-1}{n} = \frac{1}{2} \cdot \frac{1}{n} = \frac{1}{2n}$$

Hence, given that today is sunny, the probability that the next n days will all be rainy is: $\frac{1}{2n}$. ■

Problem 3

What is the probability that at least one other student at your table is from your dorm? You can assume that the number of people is large, so events of knowing different people at the party are independent.

Solution
Let $A_0 = \{0 \text{ people are from your dorm}\}, A_1 = \{1 \text{ person is from your dorm}\}, A_2 = \{2 \text{ people are from your dorm}\}$, $A_3 = \{3 \text{ people are from your dorm}\}, B = \{\text{You know all 3 of the people}\}$. Since the question is asking given that you know all 3 of the people at the table, what is the probability that at least one of the students is from your dorm, we can calculate the probability that none of them are from your dorm (which is the complement) and subtract that probability from 1. We first calculate the individual probabilities: $\mathbf{P}(A_0) = 0.99^3, \mathbf{P}(A_1) = 3 \cdot (0.01) \cdot (0.99)^2, \mathbf{P}(A_2) = 3 \cdot (0.01)^2 \cdot (0.99), \mathbf{P}(A_3) = (0.01)^3, \mathbf{P}(B) = 0.02, \mathbf{P}(B|A_0) = (0.02)^2 \cdot 0.3, \mathbf{P}(B|A_2) = (0.02) \cdot 0.3^2, \mathbf{P}(B|A_3) = 0.3^3$. The extra 3 given in $\mathbf{P}(A_1), \mathbf{P}(A_2)$ is due to the 3!/2! ways such an occurence can happen. Then we can apply Bayes' Rule,

$$\begin{aligned}\mathbf{P}(A_0|B) &= \frac{\mathbf{P}(A_0) \cdot \mathbf{P}(B|A_0)}{\mathbf{P}(A_0) \cdot \mathbf{P}(B|A_0) + \mathbf{P}(A_1) \cdot \mathbf{P}(B|A_1) + \mathbf{P}(A_2) \cdot \mathbf{P}(B|A_2) + \mathbf{P}(A_3) \cdot \mathbf{P}(B|A_3)} \\ &= \frac{0.99^3 \cdot 0.02^3}{0.99^3 \cdot 0.02^3 + 3 \cdot (0.01) \cdot (0.99)^2 \cdot (0.02)^2 \cdot 0.3 + 3 \cdot (0.01)^2 \cdot (0.99) \cdot (0.02) \cdot 0.3^2 + (0.01)^3 \cdot 0.3^3} \\ &\approx 0.6549\end{aligned}$$

Subtracting this from one, we get that given that you're sitting with 3 other people you know, the probability that at least one other student at the table is from your dorm is approximately $1 - 0.6549 = 0.3451$. ■

Problem 4

What is the probability of the event that you own the textbook for every class you enrolled?

Solution
For each of the n classes, we have that the probability we enroll is p_1, so the probability we don't enroll is $1 - p_1$. The probability that we purchase the textbook is p_2, so the probability we don't buy the textbook is $1 - p_2$. For any class, we have the following cases:

(1) we enroll and we buy the book
(2) we enroll and we don't buy the book
(3) we don't enroll and we buy the book
(4) we don't enroll and we don't buy the book

Note here that while case (1) more directly addresses the question, we include cases (3) and (4) in our calculation because there is no imposed restriction on classes we are not enrolled in. Therefore, we calculate our probability that we own the book for every in which class we're enrolled to be:

$$(p_1 p_2 + (1 - p_1)p_2 + (1 - p_1)(1 - p_2) = 1 + p_1 p_2 - p_1. \tag{1}$$

Since there are n classes, we take the probability given in (1) and multiply it by itself n times. Thus, the probability $(1 + p_1 p_2 - p_1)^n$. ■

Problem 5

The test is 95% accurate if the student is not overstressed, but only 85% accurate if the student is overstressed. 99.5% of all students are overstressed. Given that a particular student tests negative for stress, what is the probability that the test results are correct, and that this student is not overstressed.

Solution
Let A be the event that the student is not overstressed. Let B be the event that the student tests negative for stress. Hence, we are trying to find $\mathbf{P}(A|B)$. From the problem, we're given that $\mathbf{P}(B|A) = 0.95, \mathbf{P}(A) = 1 - 0.995 = 0.005, \mathbf{P}(A^c) = 0.995, \mathbf{P}(B|A^c) = 1 - 0.85 = 0.15$. Since A, A^c are disjoint events that for a partition of the sample space, and since their respective probabilities are strictly greater than 0, we can apply Bayes' Rule and see that

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A)\mathbf{P}(B|A)}{\mathbf{P}(A)\mathbf{P}(B|A) + \mathbf{P}(A^c)\mathbf{P}(B|A^c)} = \frac{0.005 \cdot 0.95}{0.005 \cdot 0.95 + 0.995 \cdot 0.15} \approx 0.031.$$

The probability that the tests results are correct given that the student tests negative for stress is approximately 0.031. ■

Problem 6

A hiker starts by taking one of n available trails, denoted, $1, \cdots, n$. An hour into the hike trail i splits into $i+1$ subtrails, only one of which leads to the hiker's destination. The hiker has no map and makes random choices of train and subtrail. What is the probability of reaching the destination?

Solution
Let A be the event that the hiker reaches the destination and B_i be the event that he takes trail i for $i = 1, \cdots, n$. It's clear that $\mathbf{P}(B_i) = \frac{1}{n}$, since there are n possible trails that he can take. We can also calculate $\mathbf{P}(A|B_i) = \frac{1}{i+1}$ since only one of the $i+1$ subtrails corresponding to each trail reaches the destination. Since B_i for $i = 1, \cdots, n$ are disjoint and form a partition of the sample space of possible subtrails, we can apply the total probability law,

$$\begin{aligned}
\mathbf{P}(A) &= \mathbf{P}(B_1)\mathbf{P}(A|B_1) + \mathbf{P}(B_2)\mathbf{P}(A|B_2) + \cdots + \mathbf{P}(A)\mathbf{P}(A|B_n) \\
&= \frac{1}{n} \cdot \frac{1}{2} + \frac{1}{n} \cdot \frac{1}{3} + \cdots + \frac{1}{n} \cdot \frac{1}{n+1} \\
&= \frac{1}{n}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right),
\end{aligned}$$

so the probability of reaching the destination is given by $\frac{1}{n}(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1})$. ■

Problem 7

A magnetic tape storing information in binary form has been corrupted, so it can only be read with bit errors. The probability that you correctly detect a 0 is 0.9, while the probability that you correctly detect a 1 is 0.85. Each digit is a 1 or a 0 with equal probability. Given that you read a 1, what is the probability this is a correct reading?

Solution
Let A be the event that it is a 1 and B be the event that we read a 1. We're given that $\mathbf{P}(A) = 0.5, \mathbf{P}(B|A) =$

Problem 7 continued on next page...

$0.85, \mathbf{P}(A^c) = 0.5$, and we can calculate $\mathbf{P}(B|A^c) = 1 - \mathbf{P}(B^c|A^c) = 1 - 0.9 = 0.1$. Since A, A^c are disjoint and form a partition of the sample space, we can apply Bayes' Rule and find

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A)\mathbf{P}(B|A)}{\mathbf{P}(A)\mathbf{P}(B|A) + \mathbf{P}(A^c)\mathbf{P}(B|A^c)} = \frac{0.5 \cdot 0.85}{0.5 \cdot 0.85 + 0.5 \cdot 0.1} \approx 0.895,$$

so the probability it is a correct reading given that we read a 1 is approximately 0.895. ■

Problem 8

Let A and B be events such that $A \subset B$. Can A and B be independent?

Solution

To show that A and B are independent, we need to show that $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$. Since $A \subset B$, then $A \cap B = A$, so $A \cap B = A$ so $\mathbf{P}(A \cap B) = \mathbf{P}(A)$. Therefore, A and B are independent if $\mathbf{P}(B) = 1$ or if $\mathbf{P}(A) = 0$. ■

Problem 9

Suppose that A, B, C are independent. Use the definition of independence to show that A and $B \cup C$ are independent.

Solution

In order to show that A and $B \cup C$ are independent, we need to show that

$$\mathbf{P}(A \cap (B \cup C) = \mathbf{P}(A) \cdot \mathbf{P}(B \cup C) \tag{2}$$

We first evaluate the LHS of (2), and we see that by De Morgan's law,

$$\begin{aligned}\mathbf{P}(A \cap (B \cup C)) = \mathbf{P}((A \cap B) \cup (A \cap C)) &= \mathbf{P}(A \cap B) + \mathbf{P}(A \cap C) - \mathbf{P}((A \cap B) \cap (A \cap C)) \\ &= \mathbf{P}(A \cap B) + \mathbf{P}(A \cap C) - \mathbf{P}(A \cap B \cap C)\end{aligned}$$

Since A, B, C are independent, we can write this as

$$\text{LHS} = \mathbf{P}(A)\mathbf{P}(B) + \mathbf{P}(A)\mathbf{P}(C) - \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)$$

Evaluating the RHS, we see

$$\begin{aligned}\mathbf{P}(A)\mathbf{P}(B \cup C) &= \mathbf{P}(A) \cdot (\mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(B \cap C)) \\ &= \mathbf{P}(A)\mathbf{P}(B) + \mathbf{P}(A)\mathbf{P}(C) - \mathbf{P}(A)\mathbf{P}(B \cap C) \\ &= \mathbf{P}(A)\mathbf{P}(B) + \mathbf{P}(A)\mathbf{P}(C) - \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)\end{aligned}$$

We can write the last equality because we know that B, C are independent. Hence, we've shown that LHS = RHS, so the equality is proven. ■

Problem 10

A parking lot contains 100 cars that all look quite nice from the outside. However, K of these cars happen to be lemons. The number K is known to lie in the range $\{0, 1, \cdots, 9\}$, with all values equally likely.

(a) We testdrive 20 distinct cars chosen at random, and to our pleasant surprise, none of them turns out ot be a lemon. Given this knowledge, what is the probability that $K = 0$?

Solution
Let

$$A_i = \{K = i\}, i = 0, \cdots, 9$$
$$B = \{\text{none of the 20 distinct cars chosen at random are lemons}\}$$

Since the A_i's form a partition of the sample space, we can apply Bayes' Rule and write:

$$\mathbf{P}(A_0|B) = \frac{\mathbf{P}(A_0) \cdot \mathbf{P}(B|A_0)}{\mathbf{P}(A_0)\mathbf{P}(B|A_0) + \mathbf{P}(A_1)\mathbf{P}(B|A_1) + \cdots + \mathbf{P}(A_9)\mathbf{P}(B|A_9)}$$

We then calculate all the individual probabilities. $\mathbf{P}(A_0) = \mathbf{P}(A_1) = \cdots = \mathbf{P}(A_9) = 1/10$. It's clear that $\mathbf{P}(B|A_0) = 1$. The rest of the conditional probabilities are calculating the probability of getting no lemons with i lemons within the 100 cars. Then, we see that

$$\mathbf{P}(B|A_1) = \binom{99}{20} \Big/ \binom{100}{20}, \mathbf{P}(B|A_2) = \binom{98}{20} \Big/ \binom{100}{20}$$
$$\mathbf{P}(B|A_3) = \binom{97}{20} \Big/ \binom{100}{20}, \mathbf{P}(B|A_4) = \binom{96}{20} \Big/ \binom{100}{20}$$
$$\mathbf{P}(B|A_5) = \binom{95}{20} \Big/ \binom{100}{20}, \mathbf{P}(B|A_6) = \binom{94}{20} \Big/ \binom{100}{20}$$
$$\mathbf{P}(B|A_7) = \binom{93}{20} \Big/ \binom{100}{20}, \mathbf{P}(B|A_8) = \binom{92}{20} \Big/ \binom{100}{20}$$
$$\mathbf{P}(B|A_9) = \binom{91}{20} \Big/ \binom{100}{20}$$

Now, we can calculate the conditional probability

$$\mathbf{P}(A_0|B) = \frac{1/10 \cdot 1}{\frac{1}{10} \cdot \left(\binom{100}{20} + \binom{99}{20} + \binom{98}{20} + \binom{97}{20} + \binom{96}{20} + \binom{95}{20} + \binom{94}{20} + \binom{93}{20} + \binom{92}{20} + \binom{91}{20} \right) \Big/ \binom{100}{20}}$$

■

(b) Repeat part (a) when the 20 cars are chosen with replacement; that is, at each testdrive, each car is equally likely to be selected, including those that were selected earlier.

Solution
Using the events as defined in part (a), we recalculate the individual conditional probabilities. Note that these are now independent events.

$$\mathbf{P}(B|A_1) = (99/100)^{20}, \mathbf{P}(B|A_2) = (98/100)^{20}, \mathbf{P}(B|A_3) = (97/100)^{20}, \mathbf{P}(B|A_4) = (96/100)^{20}$$
$$\mathbf{P}(B|A_5) = (95/100)^{20}, \mathbf{P}(B|A_6) = (94/100)^{20}, \mathbf{P}(B|A_7) = (93/100)^{20}, \mathbf{P}(B|A_8) = (92/100)^{20}$$
$$\mathbf{P}(B|A_9) = (91/100)^{20}$$

Then, we use these to calculate the original conditional probability

$$\mathbf{P}(A_0|B) = \frac{1/10 \cdot 1}{\frac{1}{10} \cdot (100^{20} + 99^{20} + 98^{20} + 97^{20} + 96^{20} + 95^{20} + 94^{20} + 93^{20} + 92^{20} + 91^{20})/100^{20}}$$

■