Self Assignment Matrix Theory (Matrices, Determinants, Linear Equations)

2024

Matrix Theory

0.1 Introduction

Matrix theory is a fundamental tool in computer science, especially in areas such as algorithms, data structures, and machine learning. In this document, we will cover the basics of matrices, determinants, and how they relate to solving systems of linear equations.

0.2 Matrices

A **matrix** is a rectangular array of numbers arranged in rows and columns. Matrices are useful in representing linear transformations and systems of linear equations.

For example, a matrix A with 2 rows and 3 columns looks like:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

The element a_{ij} represents the entry in the i-th row and j-th column.

0.2.1 Matrix Operations

The most common operations on matrices are addition, scalar multiplication, and matrix multiplication.

1. Matrix Addition: Two matrices of the same dimensions can be added by adding their corresponding elements. For matrices A and B:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}$$

2. Scalar Multiplication: Multiplying a matrix by a scalar λ involves multiplying every element of the matrix by λ:

$$\lambda A = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix}$$

3. Matrix Multiplication: The product of two matrices A and B (if defined) is given by:

$$(AB)_{ij} = \sum_k a_{ik} b_{kj}$$

For example, multiplying a 2×3 matrix with a 3×2 matrix gives a 2×2 matrix.

0.2.2 Example

Let A and B be:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

The product AB is:

$$AB = \begin{bmatrix} (1 \times 2) + (2 \times 1) & (1 \times 0) + (2 \times 2) \\ (3 \times 2) + (4 \times 1) & (3 \times 0) + (4 \times 2) \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix}$$

0.3 Determinants

The **determinant** of a square matrix is a scalar value that can be computed from its elements and encodes certain properties of the matrix, such as whether it is invertible.

For a 2×2 matrix A, the determinant is:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For a 3×3 matrix A, the determinant is:

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

0.3.1 Example

For the matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

The determinant is:

$$\det(A) = (2 \times 4) - (1 \times 3) = 8 - 3 = 5$$

0.4 Solving Systems of Linear Equations Using Matrices

A system of linear equations can be written in matrix form as:

$$A\mathbf{x} = \mathbf{b}$$

where A is the matrix of coefficients, $\mathbf{x}$ is the vector of unknowns, and $\mathbf{b}$ is the vector of constants.

For example, the system:

$$\begin{aligned} 2x + y &= 5 \\ 3x + 4y &= 6 \end{aligned}$$

can be written in matrix form as:

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

0.4.1 Solving by Inverse Matrix Method

The solution is found by multiplying both sides by the inverse of matrix A, provided that A is invertible:

$$\mathbf{x} = A^{-1}\mathbf{b}$$

where A^{-1} is the inverse of A.

0.4.2 Example

For the system:

$$\begin{aligned} 2x + y &= 5 \\ 3x + 4y &= 6 \end{aligned}$$

we first compute the inverse of A:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

The determinant of A is:

$$\det(A) = (2 \times 4) - (1 \times 3) = 8 - 3 = 5$$

The inverse of A is:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$$

Multiplying A^{-1} by $\mathbf{b}$ gives the solution:

$$\mathbf{x} = \frac{1}{5} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 14 \\ -3 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -0.6 \end{bmatrix}$$

0.5 Conclusion

Matrix theory is an essential mathematical tool in computer science, with applications ranging from solving systems of equations to modeling complex transformations in graphics and machine learning. A solid understanding of matrix operations, determinants, and their use in solving linear equations is critical for tackling more advanced topics in algorithms, data science, and artificial intelligence.