Self Assignment: Numerical Solution

2024

Introduction

Numerical solutions are essential for solving mathematical problems that cannot be solved analytically or have complex solutions. In Computer Science, numerical methods are used extensively for solving systems of linear and nonlinear equations, numerical integration, and in areas such as machine learning, simulations, and optimizations.

This self assignment will cover solving linear and nonlinear equations, and numerical integration.

1 Solving Equations

Numerical methods are often required for solving equations, particularly when analytical solutions are difficult or impossible. These methods are divided into:

- **Linear Equations**
- **Nonlinear Equations**

1.1 Linear Equations

Linear equations involve finding the solutions to systems of equations that can be represented in matrix form:

$$Ax = b$$

Where:

- A is a square matrix of coefficients.
- x is the vector of unknowns.
- b is the vector of constants.

2 Gaussian Elimination

Gaussian elimination is a method used to solve systems of linear equations. It involves performing operations on the rows of a matrix to bring it into a form that is easier to solve. Specifically, we transform the system into an upper triangular matrix, where all elements below the main diagonal are zeros.

2.1 Basic Steps

The process of Gaussian elimination consists of the following steps:

1. Convert the system of linear equations into an augmented matrix.
2. Use row operations to eliminate the variables step by step, transforming the matrix into upper triangular form.
3. Once in upper triangular form, use *back substitution* to find the solution.

Let us try break it down each step with a simple example.

2.2 Example: Solving a System of Equations

Consider the following system of two equations:

$$\begin{aligned} 2x + 3y &= 5 \\ 4x + 6y &= 10 \end{aligned}$$

This system can be written in matrix form as:

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

The augmented matrix for this system is:

$$\begin{pmatrix} 2 & 3 & | & 5 \\ 4 & 6 & | & 10 \end{pmatrix}$$

Step 1: Eliminate the First Variable

Our goal is to eliminate the first variable from the second equation. We do this by making the first element of the second row zero.

To achieve this, subtract $2\times$ (Row 1) from Row 2. That is, perform the operation:

$$R_2 \to R_2 - 2 \times R_1$$

Carrying out this operation gives:

$$\begin{pmatrix} 2 & 3 & | & 5 \\ 0 & 0 & | & 0 \end{pmatrix}$$

Notice that after performing the row operation, the second row is entirely zero. This indicates that the system is *dependent*, meaning it has infinitely many solutions.

Step 2: Interpreting the Result

The matrix now looks like this:

$$\begin{pmatrix} 2 & 3 & | & 5 \\ 0 & 0 & | & 0 \end{pmatrix}$$

The first row corresponds to the equation:

$$2x + 3y = 5$$

The second row corresponds to:

$$0 = 0$$

Since the second equation is always true (it contains no information about x or y), the system has infinitely many solutions. This means the values of x and y are not uniquely determined.

Step 3: Parametrizing the Solutions

Because we have one equation and two unknowns, we can express one variable in terms of the other. Solving the first equation for x, we get:

$$2x + 3y = 5 \quad \Rightarrow \quad x = \frac{5 - 3y}{2}$$

Therefore, the solution can be written as:

$$x = \frac{5}{2} - \frac{3}{2}y$$

Here, y can take any value, and x depends on y. This gives us an infinite set of solutions, where y is free to vary.

2.3 Key Concepts

Let's summarize the key concepts of Gaussian elimination:

- **Row Operations:** You can perform three types of row operations:
  1. Swap two rows.
  2. Multiply a row by a non-zero scalar.
  3. Add or subtract multiples of one row from another row.
- **Upper Triangular Form:** The goal of Gaussian elimination is to transform the matrix into upper triangular form, where all the elements below the diagonal are zeros.

- **Back Substitution:** Once the matrix is in upper triangular form, the system of equations can be solved by back substitution, starting from the last row and working upwards.
- **Dependent Systems:** If you obtain a row of all zeros (as in this example), it indicates that the system has either infinitely many solutions or no solutions, depending on the other rows.

3 Nonlinear Equations

Nonlinear equations are more challenging to solve compared to linear equations, as they do not have a straightforward solution method. A common numerical approach for solving a single nonlinear equation $f(x) = 0$ is **Newton's Method**, which iteratively finds an approximate solution.

3.1 Newton's Method

Newton's method is an iterative process used to find roots of nonlinear equations. Starting from an initial guess x_0, the next approximation is computed using the following formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This process is repeated until the values of x_n converge to a sufficiently accurate solution. The function $f(x)$ and its derivative $f'(x)$ must be known for the method to work effectively.

3.1.1 Example:

Consider the nonlinear equation:

$$f(x) = x^2 - 2$$

We aim to find the square root of 2. First, compute the derivative:

$$f'(x) = 2x$$

Using an initial guess of $x_0 = 1.5$, the next approximation x_1 is calculated as:

$$x_1 = 1.5 - \frac{1.5^2 - 2}{2 \cdot 1.5} = 1.4167$$

Repeating the process with this value as the new guess:

$$x_2 = 1.4167 - \frac{1.4167^2 - 2}{2 \cdot 1.4167} \approx 1.4142$$

After a few iterations, the method converges to $x = 1.4142$, which is a good approximation for $\sqrt{2} \approx 1.4142$.

4 Numerical Integration

Numerical integration is used when the integral of a function cannot be computed analytically. It provides approximate solutions by breaking the area under the curve into small, manageable pieces. Two common methods for numerical integration are the **Trapezoidal Rule** and **Simpson's Rule**.

4.1 Trapezoidal Rule

The Trapezoidal Rule approximates the integral by dividing the area under the curve into trapezoids and summing their areas. The formula for approximating the definite integral $\int_a^b f(x)\,dx$ is:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\,(f(a)+f(b))$$

This method is simple to use, though its accuracy depends on the number of points used.

4.1.1 Example:

We will approximate the integral:

$$\int_0^1 x^2\,dx$$

Using the Trapezoidal Rule with $a = 0$, $b = 1$, and $f(x) = x^2$, the approximation is:

$$\frac{1-0}{2}\,(f(0)+f(1)) = \frac{1}{2}\left(0^2+1^2\right) = \frac{1}{2}$$

The exact value of the integral is:

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

As we can see, the Trapezoidal Rule gives an approximation of $\frac{1}{2}$, which is close to the exact result, but not exact.

4.2 Simpson's Rule

Simpson's Rule provides a more accurate approximation by using parabolic segments instead of straight lines to approximate the curve. The formula for Simpson's Rule is:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

Simpson's Rule is generally more accurate than the Trapezoidal Rule, especially when the function being integrated is well-approximated by a quadratic.

4.2.1 Example:

We will now apply Simpson's Rule to approximate the same integral:

$$\int_0^1 x^2 \, dx$$

First, we calculate:

$$f\left(\frac{0+1}{2}\right) = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Now using Simpson's Rule:

$$\int_0^1 x^2 \, dx \approx \frac{1-0}{6}\left(f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right)$$

$$= \frac{1}{6}\left(0 + 4 \times \frac{1}{4} + 1\right) = \frac{1}{6} \times (1+1) = \frac{1}{3}$$

In this case, Simpson's Rule gives the exact value of the integral $\frac{1}{3}$.

5 Conclusion

So in Conclusion, numerical methods are crucial for approximating solutions to equations and integrals when analytical methods are impractical or impossible. Understanding these techniques is essential for anyone studying computational fields like computer science. Methods such as Gaussian elimination, Newton's method, and numerical integration (Trapezoidal and Simpson's Rules) provide the tools needed to handle a wide variety of real-world problems.